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EQUILIBRIUM BASED ITERATIVE SOLUTIONS FOR THE NON-LINEAR BEAM PROBLEM

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SUMMARY

The paper describes a procedure for the non-linear analysis of structures which are an assemblage of beams with material non-linearities of general type; the approach uses the equilibrium integrals and a consistent iterative formulation at the element level, within the general framework of the displacement method for the solution of the global structural problem. The application of different approaches to the non-linear beam problem is presented and discussed including the traditional stiffness and flexibility approach and some mixed formulations. The proposed equilibrium-based approach is shown to be more accurate and more robust than the traditional compatibility-based approach, on which most of the non-linear beam elements available today are based. Similar advantages are also found with respect to an approach based on the three-field mixed assumed strain method.

KEY WORDS: non-linear; finite element analysis; beam element algorithms; solution strategies; mixed approaches

1. INTRODUCTION

The paper presents a new approach to the numerical, static or dynamic, analysis of structures, made up of an assemblage of non-linear beam elements with spread material non-linearities. An essential step in the numerical incremental solution of such a problem, within the usual framework of the displacement method at the global level, consists in finding, for each individual element, the end forces which correspond to given end displacements, together with the associated generalized strains and stresses at the different sections along the element; this step is often referred to as '*element state determination*'. For simplicity, in the following, the beam element will be considered as partially constrained at the ends, so that no rigid-body displacement is allowed, and, as a consequence, we shall refer to end deformations, $\mathbf{Q}$, rather than to end displacements and to the associated end forces (or generalized stresses), $\mathbf{P}$.

The standard stiffness and flexibility approaches will be briefly reviewed before going into the presentation of the proposed algorithms. After that, a review of some mixed approaches is carried out, showing the application of *two*- and *three-field* formulation to the non-linear beam problem, pointing out differences and similarities with the proposed algorithms.

Finally, numerical examples are presented so as to compare the behaviour of the various approaches when implemented in a non-linear finite element beam used in a standard, displacement based, Finite Element (FE) code.

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2. SOLUTION STRATEGY REVIEW

The standard *stiffness approach* to the solution of the element state determination is based on the following.

(a) Assuming predefined continuous deformation functions along the beam ($\mathscr{B}$), which are compatible with the given end deformations:

$$\Delta \mathbf{q}(\xi) = \mathbf{a}(\xi)\Delta \mathbf{Q} \tag{1}$$

where $\mathbf{q}(\xi)$ is a vector of n^{s} strain components, $\mathbf{a}(\xi)$ a matrix of $n^{\mathrm{s}} \times n^{\mathrm{d}}$ shape functions and $\mathbf{Q}$ is the vector of the element deformation.

(b) Obtaining from the section deformations the section generalized stresses using a constitutive equation in the form $\mathbf{p}(\xi) = \mathbf{p}[\mathbf{q}(\xi)]$ or in the equivalent incremental linearized form :

$$\Delta \mathbf{p}(\xi) = \mathbf{k}(\xi)\Delta \mathbf{q}(\xi) + \mathbf{r}_p(\xi) \tag{2}$$

where $\mathbf{p}(\xi)$ is a vector of n^{s} stress components, $\mathbf{k}(\xi)$ is the local stiffness matrix and $\mathbf{r}_p(\xi)$ is the vector of the local stress residuals.

(c) Imposing equilibrium via the application of the virtual displacement principle so as to obtain the element end forces $\Delta \mathbf{P}$:

$$\Delta \mathbf{P} = \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\mathbf{k}(\xi)\mathbf{a}(\xi)\,\mathrm{d}\xi\,\Delta \mathbf{Q} + \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\mathbf{r}_p(\xi)\,\mathrm{d}\xi = \mathbf{K}\,\Delta \mathbf{Q} + \mathbf{R}_p \tag{3}$$

together with the definition of the element stiffness matrix $\mathbf{K}$, and of the end residual force vector $\mathbf{R}_p$.

This approach is still today the most widely used, because it gives very straightforwardly, and without element iterations, the solution of the *element state determination* step. It presents some drawbacks though: equilibrium is satisfied only in an integral sense over the element, but not locally at the different sections along the beam; the predefined deformation shape functions $\mathbf{a}(\xi)$ do not correspond to the exact solution of the beam problem, except for some special cases, and do not adapt themselves to follow the stiffness modifications which occur because of the non-linear behaviour. And yet, the non-linear behaviour must be monitored to obtain the section forces (2), a step which is likely to represent most of the computational demand when a sophisticated local constitutive behaviour is used.

The corresponding dual *flexibility approach* can be used directly only with static boundary conditions. In this case it consists of:

(i) Expressing the element generalized stress field as a function of the given element end forces $\Delta \mathbf{P}$:

$$\Delta \mathbf{p}(\xi) = \mathbf{b}(\xi)\Delta \mathbf{P} \tag{4}$$

where $\mathbf{b}(\xi)$ is a matrix containing $n^{\mathrm{s}} \times n^{\mathrm{d}}$ equilibrium integrals.

(ii) Obtaining from the section generalized stresses the section deformations via the constitutive equation in the flexibility form $\mathbf{q}(\xi) = \mathbf{q}[\mathbf{p}(\xi)]$ or in the equivalent incremental linearized form

$$\Delta \mathbf{q}(\xi) = \mathbf{f}(\xi)\Delta \mathbf{p}(\xi) + \mathbf{r}_q(\xi) \tag{5}$$

where $\mathbf{f}(\xi)$ is the local flexibility matrix and $\mathbf{r}_q(\xi)$ are the local strain residuals.

(iii) Enforcing compatibility via the application of the virtual force principle so as to obtain the element end deformations

$$\Delta\mathbf{Q} = \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{f}(\xi)\mathbf{b}(\xi)\,\mathrm{d}\xi\Delta\mathbf{P} + \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{r}_q(\xi)\,\mathrm{d}\xi = \mathbf{F}\Delta\mathbf{P} + \mathbf{R}_q \tag{6}$$

together with the definition of the element flexibility matrix $\mathbf{F}$ and of the end residual deformation vector $\mathbf{R}_q$.

The flexibility approach presents the great advantage that the equilibrium integrals $\mathbf{b}(\xi)$ are exact and do not depend on the material behaviour. On the other hand, the approach needs to be adapted to the case of kinematic, rather than static, boundary conditions, as required by the problem under consideration. In view of the advantage recalled above, there has been much interest about the idea of using equilibrium integrals for the formulation and the solution of the non-linear beam problem. Many, even recent, proposals, have tried to make use of these concepts, (e.g. References 1 and 2), but they have failed to implement them in a completely consistent way and have encountered, as a consequence, different types of problems.

Consistent procedures for the formulation of equilibrium-based non-linear beam elements are illustrated in the following paragraphs. These procedures require iterations at the element level in order to solve the element state determination step, but produce, through an apparently more cumbersome approach, a very accurate and robust algorithm.

3. EQUILIBRIUM-BASED ITERATIVE SOLUTIONS

Since the element state determination step must be performed using kinematic boundary conditions (assigned nodal displacements), (6) needs to be inverted:

$$\Delta\mathbf{P} = \mathbf{F}^{-1}\Delta\mathbf{Q} - \mathbf{F}^{-1}\mathbf{R}_q \tag{7}$$

and solved using an iterative procedure in which the deformation residuals of the previous iteration are used to calculate the end force increments at the current iteration. The iterative scheme which stems from a standard Newton–Raphson-type solution of the non-linear equation (7) produces the following expressions for the end force increments and for the incremental strain field at the iteration step i:

$$\Delta\mathbf{P}_i = \Delta\mathbf{P}_{i-1} - \mathbf{F}_{i-1}^{-1} \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{r}_{q,i-1}(\xi)\,\mathrm{d}\xi \tag{8}$$

$$\Delta\mathbf{q}_i(\xi) = \Delta\mathbf{q}_{i-1}(\xi) + \mathbf{r}_{q,i-1}(\xi) - \mathbf{f}_{i-1}(\xi)\mathbf{b}(\xi)\mathbf{F}_{i-1}^{-1} \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{r}_{q,i-1}(\xi)\,\mathrm{d}\xi \tag{9}$$

The deformation field increment at step i (i.e. $\Delta\mathbf{q}_i(\xi) - \Delta\mathbf{q}_{i-1}(\xi)$ as in (9)) can be obtained substituting the nodal force increment at the same step ($\Delta\mathbf{P}^i - \Delta\mathbf{P}^{i-1}$) as found with (8) into (4) and again into (5).

When $i = 1$, (7) gives, ignoring the residuals which are not known in advance,

$$\Delta\mathbf{P}_1 = \mathbf{F}_0^{-1}\Delta\mathbf{Q} \tag{10}$$

substituting (10) in (4) and again in (5) we obtain for the strain field along the beam,

$$\Delta\mathbf{q}(\xi)_1 = \mathbf{f}_0(\xi)\mathbf{b}(\xi)\mathbf{F}_0^{-1}\Delta\mathbf{Q} \tag{11}$$

A number of different procedures based on (11) has been already used by various authors (see References 1 and 2), but the iterative solution and the problem of the residuals were not treated

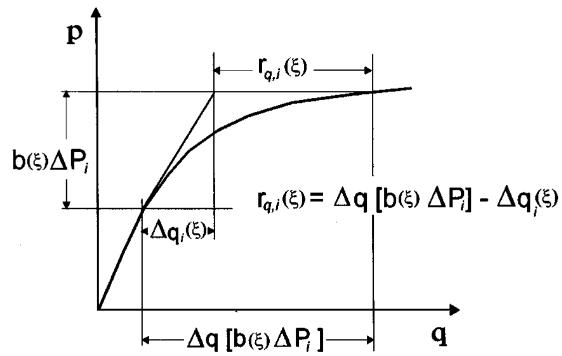

Figure 1. Local constitutive behaviour: flexibility format

consistently. In Reference 3 to get around the problem, only multilinear constitutive relations were used and an *event to event* solution strategy proposed.

For a finite load step and a continuous non-linear section behaviour instead, section deformation residuals will arise which, according to (5) and Figure 1, can be written as

$$\mathbf{r}_{q,i}(\xi) = \Delta\mathbf{q}\,[\mathbf{b}(\xi)\Delta\mathbf{P}_i] - \Delta\mathbf{q}_i(\xi) \tag{12}$$

where $\Delta\mathbf{q}_i(\xi)$ is the strain field found with (9) and $\Delta\mathbf{q}\,[\mathbf{b}(\xi)\Delta\mathbf{P}_i]$ is the strain field associated via the constitutive relation to the stress field in equilibrium with the nodal forces found with (8). The iterations stop when a norm of the second term on the right-hand side of (8), which has the meaning of vector of residual end forces, becomes less than a specified tolerance.

The element iterations procedure presented in a more general form here can already be found in a work by Ciampi and Carlesimo.[4]

It is interesting to observe that the constitutive relation does not need to be in the form of (5), in order to make the methodology applicable. In fact, more common are the cases in which the natural formulation for the local constitutive equation is rather in the stiffness format $\mathbf{p}(\xi) = \mathbf{p}[\mathbf{q}(\xi)]$ as in (2). In this case the exact deformation residuals $\mathbf{r}_q(\xi)$, needed in (8) and (9) for the satisfaction of the local constitutive behaviour, could still be found via a standard iterative procedure to be performed as a nested loop inside each element iteration.

But in the linearization process used for the solution of (7) there is no need for this nested loop on the local constitutive behaviour, since a linearization of the deformation residuals yield the same accuracy and a formal simplification of the governing element equations.

Expanding the residuals in Taylor's series and truncating them to the linear terms (see Figure 2) yields

$$\mathbf{r}_q(\xi) = -\mathbf{k}^{-1}(\xi)\mathbf{r}_p(\xi) \tag{13}$$

Substituting (13) into (8) and (9) yields the two modified expressions

$$\Delta\mathbf{P}_i = \Delta\mathbf{P}_{i-1} + \mathbf{F}_{i-1}^{-1}\int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{k}_{i-1}^{-1}(\xi)\mathbf{r}_{p,i-1}(\xi)\,\mathrm{d}\xi \tag{14}$$

$$\Delta\mathbf{q}(\xi)_i = \Delta\mathbf{q}(\xi)_{i-1} - \mathbf{k}_{i-1}^{-1}(\xi)\mathbf{r}_{p,i-1}(\xi) + \mathbf{k}_{i-1}^{-1}(\xi)\mathbf{b}(\xi)\mathbf{F}_{i-1}^{-1}\int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{k}_{i-1}^{-1}(\xi)\mathbf{r}_{p,i-1}(\xi)\,\mathrm{d}\xi \tag{15}$$

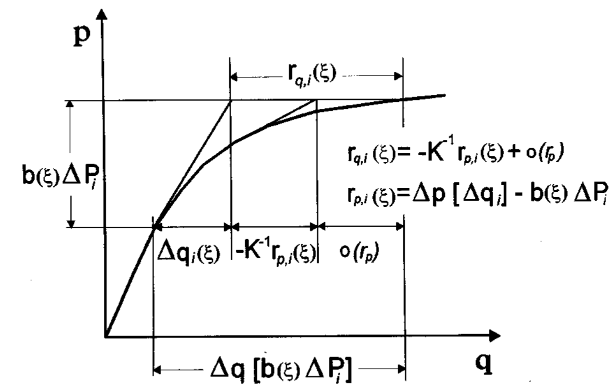

Figure 2. Local constitutive behaviour: Stiffness format

In (14) and (15) $\mathbf{F}$ and $\mathbf{r}_p(\xi)$ are determined, respectively, as

$$\mathbf{F} = \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{k}^{-1}(\xi)\mathbf{b}(\xi)\,\mathrm{d}\xi \tag{16}$$

$$\mathbf{r}_{p,i}(\xi) = \Delta\mathbf{p}\left[\Delta\mathbf{q}_{i-1}(\xi)\right] - \mathbf{b}(\xi)\Delta\mathbf{P}_i \tag{17}$$

Substitution of (17) into (14) and (15) yields two equivalent expressions, of much simpler form

$$\begin{aligned} \Delta\mathbf{P}_i &= \mathbf{F}_{i-1}^{-1} \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{k}_{i-1}^{-1}(\xi)\Delta\mathbf{p}\left[\Delta\mathbf{q}_{i-1}(\xi)\right]\mathrm{d}\xi \\ \Delta\mathbf{q}_i(\xi) &= \Delta\mathbf{q}_{i-1}(\xi) + \mathbf{k}_{i-1}^{-1}(\xi)\left\{\mathbf{b}(\xi)\Delta\mathbf{P}_i - \Delta\mathbf{p}\left[\Delta\mathbf{q}_{i-1}(\xi)\right]\right\} \end{aligned} \tag{18}$$

Equation $(18)_1$ shows that the nodal forces at a given step are a special form of weighted integrals of the internal element forces corresponding to the strain field found in the previous iteration. Equation $(18)_2$ states instead that in each iteration the correction to the strain field is found as the product of the local flexibility matrix $\mathbf{k}^{-1}(\xi)$ times the difference between the balanced internal forces at the given step $\mathbf{b}(\xi)\Delta\mathbf{P}_i$ and the forces corresponding to the strain field at the previous step $\Delta\mathbf{p}\left[\Delta\mathbf{q}_{i-1}(\xi)\right]$.

The procedure which stems out of the above equations is the following:

(1) The sections forces corresponding to the strain field given by (11), compatible with the assigned end deformations, are calculated at the integrations points along the element $\Delta\mathbf{p}(\xi) = \Delta\mathbf{p}[\Delta\mathbf{q}(\xi)]$.
(2) The integral in $(18)_1$ is computed using, for example, the Gauss's quadrature scheme, and the element nodal forces are found.
(3) A new approximation for the element strain field is found at each integration point, according to $(18)_2$.
(4) If a selected norm of the second term on the right-hand side of (14) is not less then a specified tolerance, the cycle is repeated.

The algorithm in the form given by (18) has been formulated and used for the first time in a fibre element by Petrangeli[5] (see also Petrangeli and Ciampi[6]).

A further understanding of the proposed algorithms can be achieved as follows. Let us premultiply both sides of (9) and $(18)_2$ by $\mathbf{b}^{\mathrm{T}}(\xi)$ and integrate over the element; the second terms on the right hand side vanish which implies that we can write the strain field in a compact notation as

$$\boxed{\Delta\mathbf{q}(\xi) = \mathbf{a}(\xi)\Delta\mathbf{Q} + \sum_{i=1}^{n} \Delta\tilde{\mathscr{E}}_i^h(\xi)} \tag{19}$$

Equation (19) shows that the final strain field is made of two contributions: a particular term $\mathbf{a}(\xi)\Delta\mathbf{Q}$ which satisfies the boundary conditions plus a sum of homogeneous functions $\Delta\tilde{\mathscr{E}}_i^h(\xi)$.

The particular strain field $\mathbf{a}(\xi)$ is an 'initial guess' for the algorithm. Even though the strain field of (11) generally provides the best guess, other functions could be used as well, as long as boundary conditions are satisfied. For example, when initial elastic stiffness is used, (11) yields, for a beam of constant cross-section, the well-known cubic Hermitian polynomials.

4. MIXED APPROACHES FOR THE NON-LINEAR BEAM ELEMENT

The equilibrium-based iterative algorithms presented in the previous sections can also be cast into a two-field mixed framework where, for the beam problem, the two fields collapse into a single field, the stress field, via the complete elimination of the independent strain interpolation. These approaches are also presented as *assumed stress* methods (see e.g. Reference 7).

On the other hand, very effective are today considered some three-field formulations, which, allowing for the elimination of the independent stress field from the governing equations, collapse to two-field approximations in terms of displacement and *enhanced strain* field. These formulations are also presented as *assumed strain* methods (see References 8 and 9).

The peculiarity of the beam problem is that the assumed stress methods are notably more effective than the assumed strain ones although the latter have been investigated in more detail than the former, given their superiority in other FE applications.

4.1. Two-field mixed formulation

The application of the '*two-field mixed approach*' to the beam element with assigned nodal displacements and zero body forces applied along the element will be briefly discussed. A more detailed treatment of this mixed formulation can be found in References 10 and 11.

Writing a weighted form of the constitutive equation for a finite load increment, where the weighting function is the stress field along the element $\mathbf{p}(\xi) = \mathbf{b}(\xi)\mathbf{P}$ yields

$$\mathbf{P}^{\mathrm{T}} \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi) \left\{ \Delta\mathbf{q}(\xi) - \left[\mathbf{f}(\xi)\Delta\mathbf{p}(\xi) + \mathbf{r}_q(\xi) \right] \right\} \mathrm{d}\xi = \mathbf{0} \tag{20}$$

if substitution is made of (1) for the strain field and of (4) for the stress field and taking into account that for the Virtual Work Principle $\int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{a}(\xi) = \mathbf{I}$, the previous equation can be rewritten as

$$\Delta\mathbf{Q} - \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{f}(\xi)\mathbf{b}(\xi)\,\mathrm{d}\xi\Delta\mathbf{P} - \int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\mathbf{r}_q(\xi)\,\mathrm{d}\xi = \mathbf{0} \tag{21}$$

This equation is identical to (6) and therefore yields the same algorithm described above. This iterative procedure corrects the residual of the constitutive behaviour using the equilibrium integrals which are exact. For this reason it has been preferred to classify this algorithm as *equilibrium based* since the final outcome of it depends only on the equilibrium integrals and does not contain any approximation of the constitutive behaviour due to predefined assumption of the element strain field.

4.2. Three-field mixed formulation

The application of a three-field mixed formulation to the non-linear beam element will be briefly discussed in what follows with special reference to the work of Simo and Rifai.[8]

Let us start with a strain field as in (19) where the enhanced strain field component is explicitly written as a function of a vector of parameters $\mathbf{\Lambda}$:

$$\boxed{\mathbf{q}(\xi) = \mathbf{q}^0(\xi) + \tilde{\mathscr{E}}^h(\xi) = \mathbf{a}(\xi)\mathbf{Q} + \boldsymbol{\lambda}(\xi)\mathbf{\Lambda}} \tag{22}$$

where $\boldsymbol{\lambda}(\xi)$ is a matrix of $n^{\mathrm{s}} \times n^{\tilde{\mathscr{E}}^h}$ prescribed functions which define the enhanced strain interpolation.

Then we require the *enhanced strain field* $\tilde{\mathscr{E}}^h(\xi) = \boldsymbol{\lambda}(\xi)\mathbf{\Lambda}$ to satisfy the following conditions:

Condition (i): The functions $\lambda_{i,j}(\xi)$ and $\lambda_{i,k}(\xi)$ are linearly independent for any $i = 1, 2, \ldots, n^{\mathrm{strs}}$ and $j, k = 1, 2, \ldots, n^{\tilde{\mathscr{E}}^h}$.

Condition (ii): The enhanced strain field $\tilde{\mathscr{E}}^h(\xi) = \boldsymbol{\lambda}(\xi)\mathbf{\Lambda}$ and the standard strain interpolation $\mathbf{q}^{\mathrm{o}}(\xi) = \mathbf{a}(\xi)\mathbf{Q}$ are independent.

Condition (iii): The enhanced strain field $\tilde{\mathscr{E}}^h(\xi) = \boldsymbol{\lambda}(\xi)\mathbf{\Lambda}$ and the stress field interpolation $\mathbf{p}(\xi) = \mathbf{b}(\xi)\mathbf{P}$ are orthogonal in the sense that

$$\int_{\mathscr{B}} \mathbf{b}^{\mathrm{T}}(\xi)\boldsymbol{\lambda}(\xi)\,\mathrm{d}\xi = \mathbf{0} \tag{23}$$

Writing a weighted form of the constitutive equations for a finite load increment, where the weighting function is now the strain field along the element $\mathbf{q}(\xi) = \mathbf{q}^{\mathrm{o}}(\xi) + \tilde{\mathscr{E}}^h(\xi)$ yields

$$\begin{aligned} &\mathbf{Q}^{\mathrm{T}} \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\left\{\Delta\mathbf{p}(\xi) - \left[\mathbf{k}(\xi)\Delta\mathbf{q}(\xi) + \mathbf{r}_p(\xi)\right]\right\}\,\mathrm{d}\xi \\ &+\mathbf{\Lambda}^{\mathrm{T}} \int_{\mathscr{B}} \lambda^{\mathrm{T}}(\xi)\left\{\Delta\mathbf{p}(\xi) - \left[\mathbf{k}(\xi)\Delta\mathbf{q}(\xi) + \mathbf{r}_p(\xi)\right]\right\}\,\mathrm{d}\xi = \mathbf{0} \end{aligned} \tag{24}$$

Substituting now the independent stress and strain fields, (4) and (22), and taking into account Condition (iii) the following set of equations is found:

$$\boxed{\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{a\lambda} \\ \mathbf{K}_{\lambda a} & \mathbf{K}_{\lambda\lambda} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{Q} \\ \Delta\mathbf{\Lambda} \end{Bmatrix} = \begin{Bmatrix} \Delta P + \mathbf{R}_a \\ \mathbf{R}_\lambda \end{Bmatrix}} \tag{25}$$

where

$$\begin{aligned} \mathbf{K}_{aa} &= \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\mathbf{k}(\xi)\mathbf{a}(\xi)\,\mathrm{d}\xi \\ \mathbf{K}_{\lambda\lambda} &= \int_{\mathscr{B}} \boldsymbol{\lambda}^{\mathrm{T}}(\xi)\mathbf{k}(\xi)\boldsymbol{\lambda}(\xi)\,\mathrm{d}\xi \\ \mathbf{K}_{\lambda a}^{\mathrm{T}} = \mathbf{K}_{a\lambda} &= \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\mathbf{k}(\xi)\boldsymbol{\lambda}(\xi)\,\mathrm{d}\xi \\ \mathbf{R}_a &= \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\mathbf{r}_p(\xi)\,\mathrm{d}\xi \\ \mathbf{R}_\lambda &= \int_{\mathscr{B}} \boldsymbol{\lambda}^{\mathrm{T}}(\xi)\mathbf{r}_p(\xi)\,\mathrm{d}\xi \end{aligned}$$

The residuals are in this case given by the expression

$$\mathbf{r}_p(\xi) = \Delta\mathbf{p}\left[\mathbf{a}(\xi)\Delta\mathbf{Q} + \boldsymbol{\lambda}(\xi)\Delta\boldsymbol{\Lambda}\right] - \mathbf{k}(\xi)\left[\mathbf{a}(\xi)\Delta\mathbf{Q} + \boldsymbol{\lambda}(\xi)\Delta\boldsymbol{\Lambda}\right] \tag{26}$$

Substitution of (26) into (25) and consistent formulation of the local and element stiffness matrices yield the following simple expressions to be used for the corresponding iterative algorithm implementation:

$$\begin{aligned} \Delta\boldsymbol{\Lambda}_i &= \Delta\boldsymbol{\Lambda}_{i-1} + \mathbf{K}_{\lambda\lambda}^{-1}\int_{\mathscr{B}} \boldsymbol{\lambda}^{\mathrm{T}}(\xi)\Delta\mathbf{p}\left[\Delta\mathbf{q}_{i-1}(\xi)\right]\mathrm{d}\xi \\ \Delta\mathbf{q}_i(\xi) &= \mathbf{a}(\xi)\Delta\mathbf{Q} + \boldsymbol{\lambda}(\xi)\Delta\boldsymbol{\Lambda}_i \\ \Delta\mathbf{P}_i &= \int_{\mathscr{B}} \mathbf{a}^{\mathrm{T}}(\xi)\Delta\mathbf{p}\left[\Delta\mathbf{q}_i(\xi)\right]\mathrm{d}\xi \end{aligned} \tag{27}$$

The procedure which stems out of the above equations is the following:

(1) The section forces corresponding to the strain field given by the standard displacement functions $\Delta\mathbf{p}[\Delta\mathbf{q}(\xi)] = \Delta\mathbf{p}[\mathbf{a}(\xi)\Delta\mathbf{Q}]$ are calculated at the integrations points along the element.
(2) The integral in $(27)_1$ is evaluated using, for example, the Gauss's quadrature scheme, and a new value for the enhanced strain field multipliers $\Delta\boldsymbol{\Lambda}$ is found.
(3) A new approximation for the element strain field is calculated at each integration point, according to $(27)_2$
(4) If a selected norm of the second term on the right-hand side of $(27)_1$ is not less then a specified tolerance, the cycle is repeated using the new strain interpolation.
(5) The nodal forces $\Delta\mathbf{P}$ are calculated using $(27)_3$.

Contrary to the previous approaches, in this case, the converged solution does not satisfy punctually the constitutive behaviour, i.e. $\Delta\mathbf{p}\left[\Delta\mathbf{q}_n(\xi)\right] \neq \mathbf{b}(\xi)\Delta\mathbf{P}_n$.

5. NUMERICAL IMPLEMENTATION

The iterative procedure defined by equations (8), (9) and (18) as well as (27) can be implemented in different ways depending mainly on the solution strategy adopted at the global level (i.e. the FE solver).

The available options concern the *stiffness reformulation* and the *element variables update*; the two problems are independent, and this holds true also for the algorithms presented in the paper.

The stiffness and flexibility matrices to be used in (8), (9) and (14)–(18) as well as in (27) need not to be kept constant during the element iterations, as is indicated by the subscript i which appears explicitly: by using in every iteration the new tangent values, a much faster convergence is achieved with only minor numerical burden. The element matrices are of very small size and their assembly instantaneous once the local values at the integration points along the element are known as required for the calculation of the internal residuals (12), (16) and (26).

Also the stiffness values used at the global level do not need to be the same as the ones used inside the element. For example, constant stiffness solution strategies at the global level may be used together with tangent strategies at the element level. Tangent matrices provide obviously the fastest convergence, but initial or secant values can be used as well, very much the same to what is done in all the discrete non-linear problems.

The internal element variables update can also be performed in different ways. The *path-dependent* and the *path-independent* state determination are the two basic choices. Differently to the stiffness reformulation, the internal variables update must follow the global solver strategy, which means that the nodal displacements and the element strain fields must be updated at the same time.

In the *path-dependent* strategy the element variable update is performed in every global iteration inside the load step. In this case the internal element iterations (8), (9) and (18) can be truncated after the first correction (i=2). By updating the internal strain field in every global iteration, the internal corrections are summed and the local constitutive behaviour is satisfied along with the nodal equilibrium. The algorithm (8) and (9) used in conjunction with a *consistent linearization approach* (see Reference 12) was already discussed and implemented in the work of Ciampi.[4] The same strategy can be applied to the *assumed strain methods* of (25) and (27).

The obvious advantage of this strategy is to avoid a nesting of the element iterations inside the nodal one: in fact, the computational effort to achieve an exact solution for the inner non-linear system may result in a waste of accuracy if the external non-linear system is still far from the solution.

The drawback of this implementation strategy is that the variable updates are performed in every iteration thus also on non-converged states. This may yield some fictitious change in the local strain path when using constitutive laws more general than purely non-linear elastic, allowing, for example, for different loading and unloading behaviour.

Using instead the *path-independent* strategy, the variable update is performed only on converged states therefore in each iteration the load increment and all the other nodal and element variables, are initialized to the last global converged step. In this case the internal element iterations need to be carried out to a specified tolerance since the internal corrections are lost in every new global iteration (i.e. the local strain increments are initialized in every global iteration).

This approach is more consistent where the local behaviour is strongly path dependent and an update on non-converged states can modify the solution. On the other hand, the double nesting of the iteration cycles may result in longer computer time, although this is not so obvious as it may seem, and depending on the problem the full internal iteration scheme has proved to be even faster. This may happen when only a minor number of finite elements in the mesh show a strong non-linear behaviour. In this case the full element iteration scheme may save time by reducing the global iterations needed to converge.

6. NUMERICAL VERIFICATIONS

The equilibrium-based algorithms presented above have been already implemented by the authors in different beam models (see References 4–6) and also many works have been dedicated to the application of the referred assumed strain (AS) mixed approaches (see References 8 and 9) to various types of finite elements. In the present study, the focus has been put on the comparison of these different strategies, some of them already well established in the literature although not so often applied to *beams with material non-linearities*.

In this paper, in order to compare the different algorithms a simple beam element has been chosen, the source code being kept rigorously the same except for the element solution strategy to be investigated. For this reason, all the algorithms discussed above were implemented in the same plane beam element with linear elastic axial behaviour and characterized by a non-linear elastic moment curvature relationship which can be analytically expressed in both formats (2) and (5).

This beam element has been implemented in the general purpose non-linear finite element program ANSR (see Reference 13).

Four different algorithms were tested in all, the standard stiffness approach (1)–(3) and the two equilibrium based algorithms (8), (9), (18) plus the AS mixed approach (27) both with one and full iteration procedure.

For the AS mixed approach two enhanced strain field interpolations have been tried, which are based, respectively, on third and fourth-order polynomials. With reference to the choice of two $\mathbf{\Lambda}$ parameters, the two chosen interpolations are

$$\mathbf{\Lambda}^{(3)} = \begin{bmatrix} 10\xi^3 - 12\xi^2 + 3\xi \\ 10(\xi - 1)^3 - 12(\xi - 1)^2 + 3(\xi - 1) \end{bmatrix} \tag{28}$$

$$\mathbf{\Lambda}^{(4)} = \begin{bmatrix} 15\xi^4 - 20\xi^3 + 6\xi^2 \\ 15(\xi - 1)^4 - 20(\xi - 1)^3 + 3(\xi - 1)^2 \end{bmatrix} \tag{29}$$

A plot of the first row of each of the two interpolations is shown in Figure 3. The second rows are obtained from the first ones by mirroring them with respect to the beam midpoint. The enhanced strain field interpolation enriches the standard curvature shape functions $\mathbf{a}(\xi)$ containing only linear terms and provide the curvature localization at beam ends when *softening hinges* develop. It is also easy to verify that both interpolations satisfy all the previously given conditions, in particular (23).

In Figure 4 the elastic moment curvature relation adopted for the different tested strategies is plotted. The curve is based on a logarithmic expression and shows a strong softening behaviour. A second-order parabola was also implemented to check if the algorithm showed any sensitivity. Both elastic-softening and elastic-hardening behaviour were studied by simply mirroring the parabola, but no evidence could be found of any dependency of the algorithm on the particular constitutive laws adopted. Due to this, the related results have been omitted.

The results of the four versions of the equilibrium-based algorithms (i.e. equations (8), (9) and (18) performed with *one* and *full* element iteration) are always so close to one another that they have been unified in the graphical presentation and will be addressed to, in the following, as *equilibrium-based* (EB) approach.

The first example refers to the simple cantilever of Figure 5. The structure has been modelled with one, two and four elements of the same size. In all the approaches a numerical integration with five Gauss–Lobatto's points (see Reference 14) was used.

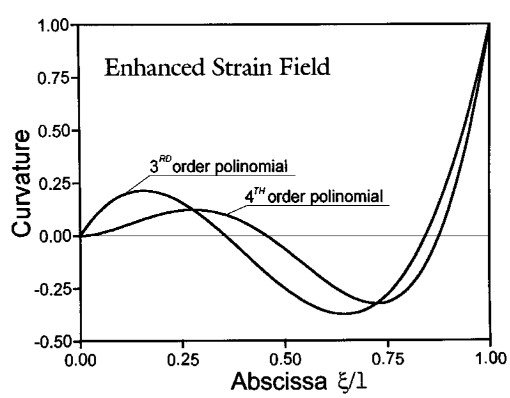

Figure 3. Enhanced shape functions used in the three-field mixed approach

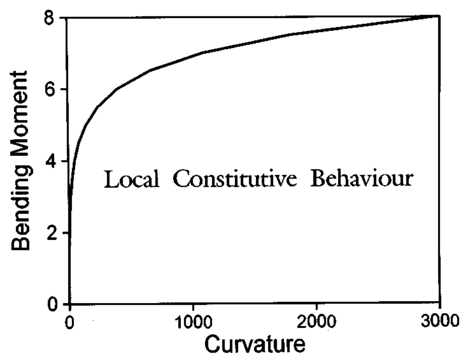

Figure 4. Moment curvature relation used in the numerical tests

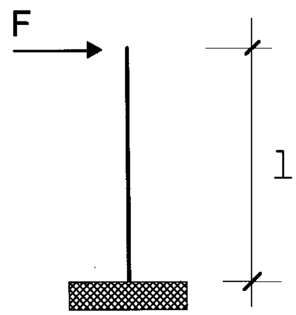

Figure 5. Cantilever: geometry, loading boundary conditions

Figure 6(a) shows the force–displacement curves for the stiffness and the EB methods. Since five Gauss–Lobatto's points provide sufficient numerical accuracy for the specific strain field, the results using the EB approach with one, two and four elements almost coincide although the one element mesh tends to sligtly overestimate the displacements, as expected from an equilibrium-based approach.

Figure 6(b) shows the results of the mixed approach for the two different enhanced strain fields; the chosen functions are able to provide the requested strain localization at the beam ends and therefore the solution is very good.

Figures 7(a)–(c) show the curvature fields along the cantilever for the different approaches at the last load step. The curves are plotted with a straight lines between the values at the integration points available from the analysis.

The curvature field for the EB approach is the same for the different meshes since the stress field is correct and the constitutive law at the integration points along the beam is always satisfied. The small differences in the force–displacement diagrams for the EB approach is due instead to the different numerical accuracy obtained when performing the integration along the cantilever with the one, two and four element meshes.

The curvature field of the AS mixed approach, where the third-order polynomial shape functions have been used, is also very accurate.

Figures 8(a)–(c) show the bending moment associated via the constitutive behaviours to the curvature fields shown previously in Figure 7.

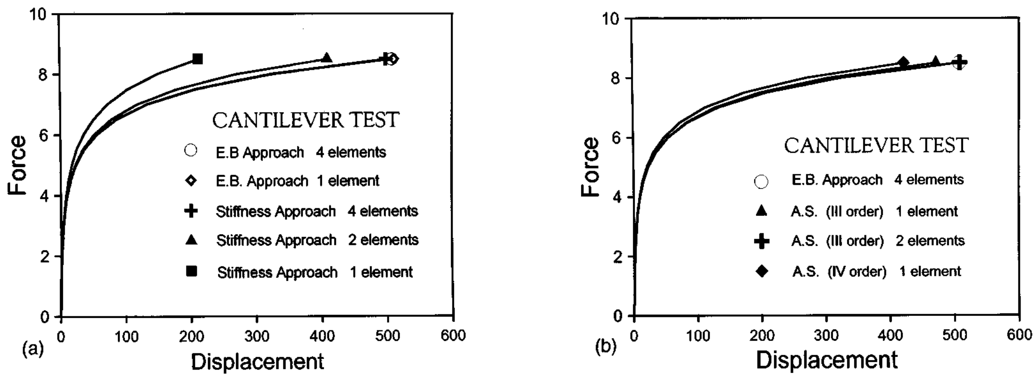

Figure 6. Cantilever: force displacement curves: (a) comparison of the stiffness and equilibrium-based approaches; (b) comparison of the equilibrium-based and assumed strain mixed approaches

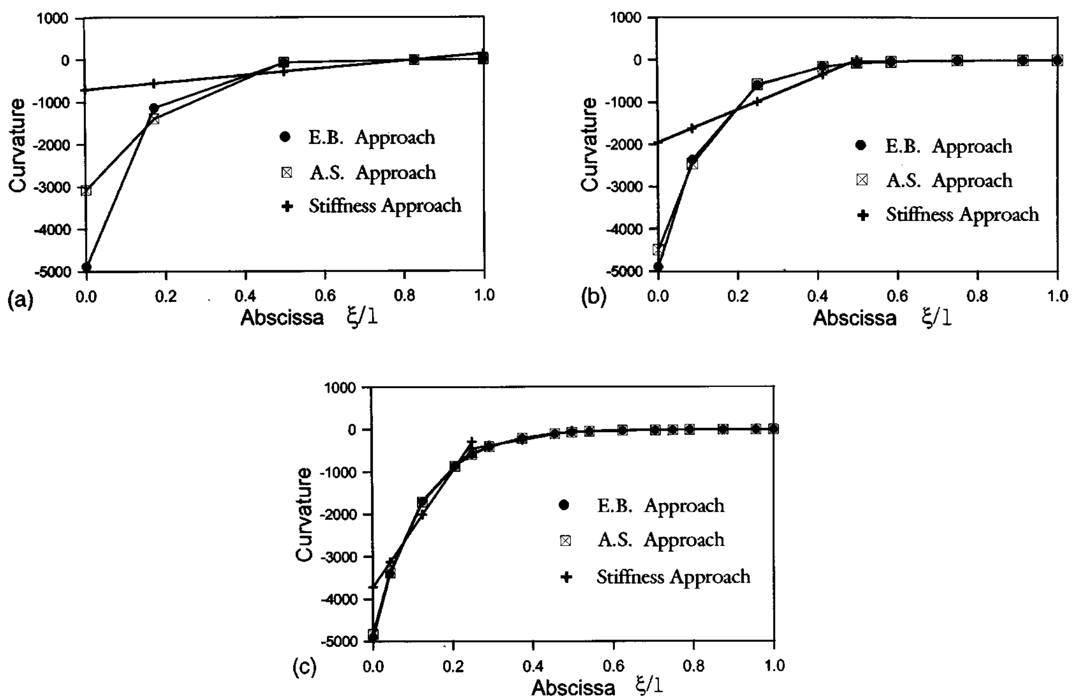

Figure 7(a)–(c). Cantilever: curvature interpolation for the different approaches. One, two and four element meshes. Gauss–Lobatto's five-point integration scheme

When analysing statically determined structures with assigned forces, the bending moments as interpolated from the nodal values are of no interest, since they are obviously in equilibrium with the external loads for all the approaches discussed.

In the EB approaches only, this bending moment interpolation does coincide with the bending moment associated, via the constitutive behaviours, to the curvature fields.

Some tests on redundant structures have been carried out using the same local constitutive behaviour as shown in Figure 4.

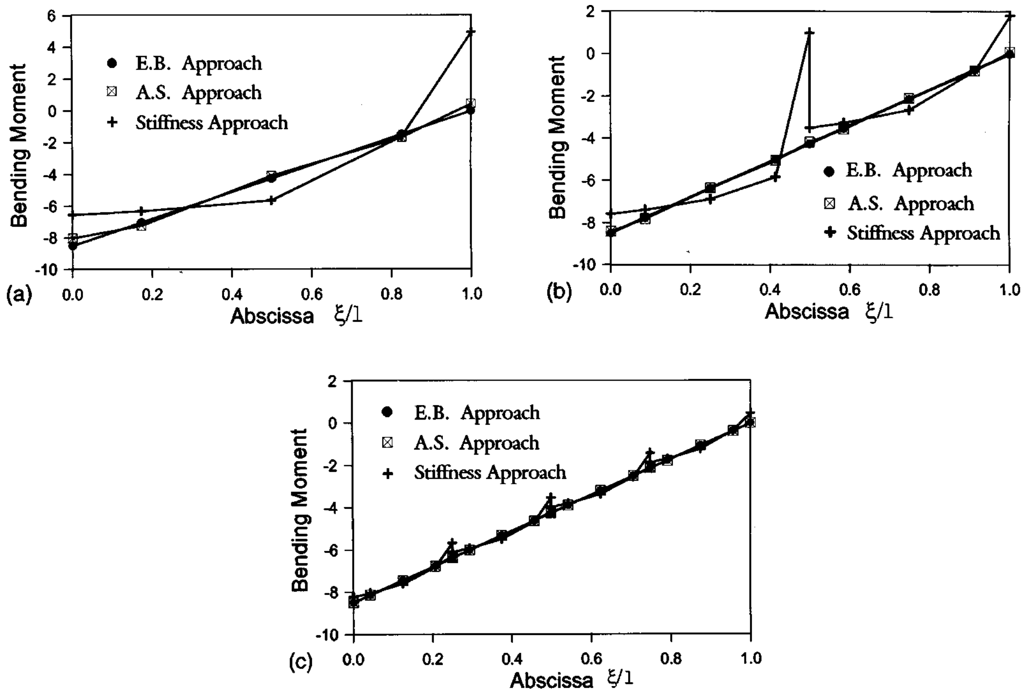

Figure 8(a)–(c). Cantilever: bending moment along the elements for the different approaches. One, two and four element meshes. Gauss–Lobatto's five-point integration scheme

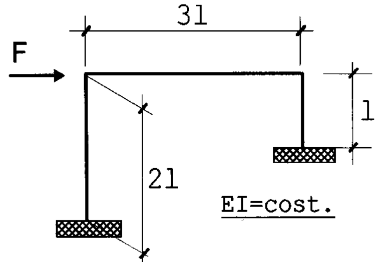

Figure 9. Asymmetric frame: Geometry, loading and boundary conditions

In Figure 10(a) the force–displacement curve for the simple frame of Figure 9 are reported comparing the stiffness and the EB approach.

Figure 10(b) shows instead the behaviour of the AS mixed approach using the two different *enhanced strain fields*. The convergence is again much better compared to the stiffness approach although is not monotonic as in the other two methods (see also the next figure).

Figure 11 summarizes the performances of the different approaches; the behaviour of the AS mixed approaches clearly depends on the enhanced strain field used in the element. Asymptotically, the results of the AS mixed approaches provide a lower bound for the displacement as in the standard stiffness formulation.

A final remark regarding the comparison of the EB and the AS approaches needs to be done. In both approaches the element strain field is found adding a particular strain field to a sum of homogeneous ones (19). In the EB approaches, both fields (11), (18_2) are found as the product of

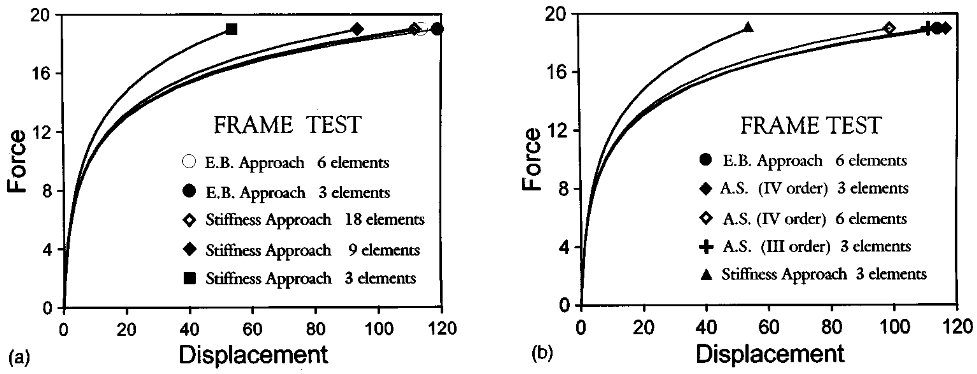

Figure 10. (a) Asymmetric frame: force–displacement curves. Comparison of the stiffness and equilibrium-based approaches; (b) comparison of the Equilibrium-based and assumed strain mixed approaches

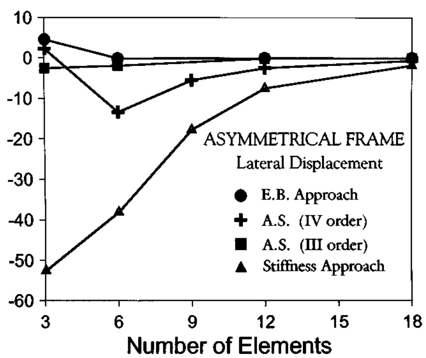

Figure 11. Convergence for the different algorithms. The true value has been assumed equal to the coincident numerical results obtained with the EB approach with 12 and 18 elements for a lateral force level of $F = 19$

a stress field times the local flexibility matrix. In the AS approaches instead, predetermined shape functions (1) and (27_2) are used.

As long as the particular strain fields are concerned, the EB approach is more efficient since (11) is always a better guess than (1); the two coincide in the linear elastic case. For the same reason the homogeneous strain fields found using the local flexibility matrix are better than any predetermined shape functions since they can adapt themselves to follow the exact distribution of the local residual along the element.

The accuracy of the strain field used in the AS instead, can only be improved by increasing the number of the homogeneous functions.

In the examples proposed above, for the AS approach, two homogeneous shape functions have been used (29). This choice yields, for the two methods, the same number of operations to perform at each element iteration since in both cases a 2×2 non-linear system has to be solved iteratively. In these examples, the accuracy of the strain field found with the EB method is much better as we can see from Figure 7(a) although, in terms of global response, the difference is smaller (see Figures 6(b) and 10(b)).

Finally, the main advantage of the EB methods, which is not evident from the results showed above, is in the number of iterations needed to converge. Because of the superiority of the equilibrium-based strain field recalled above the resulting procedure needs less iteration to achieve the equilibrium at the element level and at the nodal one.

7. CONCLUDING REMARKS

In order to provide a comprehensive framework for discussing different solution methods to the non-linear beam problem the following criteria have been adopted:

(1) All the algorithms were written in dual form such that the system primary unknowns at nodal and section level could be either generalized strains or stresses.
(2) Numerical verification for all the different algorithms was carried out on a simple beam model using the same non-linear elastic moment–curvature relation. The dual structure of the algorithms allowed for the cross-checking of the results.
(3) Numerical tests to investigate path-dependent behaviour and axial force-bending interaction have been performed on fibre model with success.

The results of this study demonstrate that the most efficient algorithms for the non-linear finite element beam are found using the equilibrium integrals, which are known independently of the local constitutive behaviour. When boundary conditions are of the kinematic type, the consistent implementation of this concept is via the iterative procedure described in the paper.

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